

# Boundary value problems and $J$ -complex curves

Alexandre Sukhov\* and Alexander Tumanov\*\*

\* Université des Sciences et Technologies de Lille, Laboratoire Paul Painlevé, U.F.R. de Mathématique, 59655 Villeneuve d'Ascq, Cedex, France, sukhov@math.univ-lille1.fr

\*\* University of Illinois, Department of Mathematics 1409 West Green Street, Urbana, IL 61801, USA, tumanov@math.uiuc.edu

Abstract. We give a solution to the problem of filling by a Levi-flat hypersurface for a class of totally real tori in  $\mathbb{C}^2$  equipped with a certain almost complex structure.

MSC: 32H02, 53C15.

Key words: almost complex structure, Levi-flat hypersurface,  $J$ -complex disc.

## 1 Introduction

The problem of gluing complex discs to real submanifolds in complex space has been a subject of extensive research. In this paper we consider the problem for a class of totally real tori in  $\mathbb{C}^2$  equipped with a certain almost complex structure. We restrict to structures  $J$  with the following characteristic property: the lines parallel to one coordinate axis are  $J$ -complex hypersurfaces. In complex dimension 2 every almost complex structure  $J$  locally has this property; we impose this condition globally. We prove (Theorem 1.1) that certain real tori can be filled by Levi-flat hypersurfaces. The result can be viewed as a solution to a Riemann-Hilbert problem for a quasi-linear elliptic equation in the plane with non-linear boundary conditions. For the usual complex structure in  $\mathbb{C}^2$ , our result was obtained earlier by Forstnerič [6] and Schnirelman [11].

Let  $(M, J)$  be an almost complex manifold. We denote by  $\mathbb{D}$  the unit disc in  $\mathbb{C}$ . We denote by  $J_{\text{st}}$  the standard complex structure of  $\mathbb{C}^n$ ; the value of  $n$  will be clear from context. A continuous map  $f : \overline{\mathbb{D}} \rightarrow M$  differentiable in  $\mathbb{D}$  is called a  $J$ -complex (or  $J$ -holomorphic) disc if it satisfies the equation  $df \circ J_{\text{st}} = J \circ df$  in  $\mathbb{D}$ . We often identify a  $J$ -complex disc  $f$  with the image  $f(\overline{\mathbb{D}})$  and call it just a disc. By the boundary of the disc  $f$  we mean the restriction  $f|_{\partial\mathbb{D}}$  which we also identify with its image.

In local coordinates  $Z = (z, w) \in \mathbb{C}^2$ , an almost complex structure  $J$  can be defined by a complex matrix  $A(Z)$  so that a map  $Z : \mathbb{D} \rightarrow U$  is  $J$ -complex if and only if it satisfies the equation

$$Z_{\bar{\zeta}} - A(Z)\overline{Z_{\zeta}} = 0. \quad (1)$$

The matrix  $A(Z)$  is defined by

$$A(Z)v = (J_{\text{st}} + J(Z))^{-1}(J_{\text{st}} - J(Z))\bar{v}. \quad (2)$$

Indeed, it is easy to see that the right-hand side of (2) is  $\mathbb{C}$ -linear in  $v \in \mathbb{C}^2$ , hence it defines a unique matrix  $A(Z)$  (see [13] for details). We call the matrix  $A$  *the complex matrix* of  $J$ . The ellipticity of (1) is equivalent to  $\det(I - A\bar{A}) \neq 0$ . In a fixed coordinate chart, the correspondence between almost complex structures  $J$  with  $\det(J_{\text{st}} + J) \neq 0$  and complex matrices with  $\det(I - A\bar{A}) \neq 0$  is one-to-one [13].

Let  $A$  be a lower triangular complex matrix function in a domain in  $\mathbb{C}^2$ . That is,

$$A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \quad (3)$$

The matrix  $A$  is the matrix of an almost complex structure if and only if  $|a| \neq 1$  and  $|c| \neq 1$ . An almost complex structure  $J$  with matrix  $A$  of the form (3) has the following characteristic property: the lines  $z = \text{const}$  are  $J$ -complex curves, that is, their tangent spaces are  $J$ -invariant.

Our main result is the following

**Theorem 1.1** *Let  $J$  be a smooth almost complex structure on  $\overline{\mathbb{D}} \times \mathbb{C}$  with matrix (3), in which  $|a| < a_0$ ,  $|c| < a_0$ , and  $0 < a_0 < 1$  is a constant. Suppose the map  $z \mapsto (z, 0)$  is  $J$ -complex, that is,  $a(z, 0) = b(z, 0) = 0$ . Suppose the first derivatives of  $a$ ,  $b$ , and  $c$  with respect to  $w$  and  $\bar{w}$  are uniformly bounded. Let  $\gamma_z \subset \mathbb{C}$  be a smooth simple closed curve depending smoothly on the parameter  $z \in b\mathbb{D}$ . Suppose that for every  $z \in b\mathbb{D}$ , the bounded component of  $\mathbb{C} \setminus \gamma_z$  contains 0. Introduce the torus  $\Lambda = \bigcup_{z \in b\mathbb{D}} \{z\} \times \gamma_z$ . Then the following hold.*

- (i) *For every point  $p \in \Lambda$  there exists a  $J$ -complex disc  $f : \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}} \times \mathbb{C}$  such that  $f(1) = p$  and whose image  $f(\overline{\mathbb{D}})$  is a graph of a non-vanishing function  $z \mapsto w(z)$  of class  $C^\infty(\overline{\mathbb{D}})$ , that is,  $f(\overline{\mathbb{D}}) = \{(z, w) : w = w(z), z \in \overline{\mathbb{D}}\}$ . In particular,  $f$  is an embedding,  $f(b\mathbb{D}) \subset \Lambda$ , and  $f(\overline{\mathbb{D}})$  does not meet  $\overline{\mathbb{D}} \times \{0\}$ . The disc  $f$  is unique up to parametrization in any Sobolev class  $W^{1,p}(\overline{\mathbb{D}})$ ,  $p > 2$ .*
- (ii) *The discs in (i) form a  $C^\infty$ -smooth one-parameter family; they depend continuously on the matrix  $A$  and the curves  $\gamma_z$ . They are disjoint and fill a smooth Levi-flat hypersurface  $\Gamma \subset \overline{\mathbb{D}} \times \mathbb{C}$  with boundary  $\Lambda$ .*
- (iii) *There is a constant  $C > 0$  that depends only on  $A$  such that if  $\gamma_z \subset r\overline{\mathbb{D}}$  for some  $r > 0$  and all  $z \in b\mathbb{D}$ , then  $\Gamma \subset \overline{\mathbb{D}} \times R\overline{\mathbb{D}}$ ,  $R = Cr$ .*

The hypothesis that the matrix  $A$  has a special form (3) is natural because it is tied to the special form of the domain  $\overline{\mathbb{D}} \times \mathbb{C}$ . In particular, it guarantees that the torus  $\Lambda$  is totally real. For a general matrix  $A$  the conclusion of the theorem fails.

As we mention above, in the integrable case ( $A = 0$ ), this result was obtained earlier by Forstnerič [6] and Schnirelman [11]. Already in this special case, Theorem 1.1 has important connections. Berndtsson and Ransford [1] used a suitable version in their proof of the corona theorem. Slodkowski [12] used similar ideas in his  $\lambda$ -lemma, which has important applications in complex dynamics and quasi-conformal maps. Recently Duval and Gayet [5] obtained a new result on gluing holomorphic discs and annuli to certain totally real tori in  $\mathbb{C}^2$ . In the integrable case, Theorem 1.1 can be used to study envelopes of holomorphy following Bedford and Klingenberg [2] and Forstnerič [6]. Another direction concerns applications in symplectic and contact geometry in the spirit of works of Gromov [7], Eliashberg [8], and others. Here Theorem 1.1 may be used in full generality, that is, for non-integrable almost complex structures. In a forthcoming paper we will use Theorem 1.1 for describing deformations of  $J$ -complex discs. Finally, a situation related to that in Theorem 1.1 often arises in the theory of holomorphic foliations, see for instance [3]. We hope that our result will find further applications in these directions and will be a useful tool in their development.

## 2 Elliptic estimates and the maximum principle

We first consider a non-homogeneous Beltrami equation

$$v_{\bar{z}} = qv_z + Q. \quad (4)$$

The following result must be well known, however we could not find a precise reference. For completeness we include a proof.

**Proposition 2.1** *(i) Let  $q$  and  $Q$  be bounded functions in  $\mathbb{D}$ ,  $|q| \leq q_0 < 1$ ,  $|Q| \leq Q_0$ , here  $q_0$  and  $Q_0$  are constants. There exists  $p > 2$  and a unique solution  $v$  of (4) in the Sobolev class  $W^{1,p}(\mathbb{D})$  with boundary conditions  $\operatorname{Re} v|_{b\mathbb{D}} = 0$ ,  $v(1) = 0$ . Furthermore  $\|v\|_{C^\alpha(\overline{\mathbb{D}})} \leq C$  with  $\alpha = \frac{p-2}{p}$ . Here  $C > 0$  and  $p$ , hence  $\alpha$ , depend on  $q_0$  and  $Q_0$  only.*

*(ii) If in addition  $\|q\|_{C^{k,\beta}(\overline{\mathbb{D}})} + \|Q\|_{C^{k,\beta}(\overline{\mathbb{D}})} \leq Q_0$  for some  $0 < \beta < 1$  and  $k \geq 0$ , then  $\|v\|_{C^{k+1,\beta}(\overline{\mathbb{D}})} \leq C$ ; here  $C > 0$  depends on  $\beta$ ,  $k$ ,  $q_0$  and  $Q_0$  only.*

The statement (ii) is not needed in this paper. We include it for completeness and future references.

**Proof :** (i) We use a classical method for solving the Beltrami equation based on the Cauchy-Green operator  $T$  and its modification ([14], Theorem 3.29; see also [4], equation (5)):

$$Tu(z) = \frac{1}{2\pi i} \int_{\mathbb{D}} \frac{u(\zeta) d\zeta \wedge d\bar{\zeta}}{\zeta - z}, \quad T_1u(z) = Tu(z) - \overline{Tu(z^*)} - 2i\operatorname{Im} Tu(1),$$

here  $z^* := 1/\bar{z}$ . Then  $v = T_1u$  solves the boundary value problem:  $v_{\bar{z}} = u$ ,  $\operatorname{Re} v|_{b\mathbb{D}} = 0$ ,  $v(1) = 0$ . We also use  $S_1u(z) = \partial_z T_1u(z)$ . The operator  $S_1$  is an isometry of  $L^2(\mathbb{D})$ .

Put  $u := v_{\bar{z}}$ . Then we have  $v = T_1 u$  and  $v_z = S_1 u$ . Then (4) turns into

$$u = qS_1 u + Q. \quad (5)$$

Since  $|q| \leq q_0 < 1$  and  $\|S_1\|_{L^2} = 1$ , then for  $p > 2$  close to 2, we have  $\|qS_1\|_{L^p} < 1$ . Hence (5) has a unique solution  $u \in L^p(\mathbb{D})$ . Then  $v = T_1 u \in W^{1,p}(\mathbb{D})$ . Then  $v \in C^\alpha$ ,  $\alpha = \frac{p-2}{p}$ . The solution is unique in  $W^{1,p}$  and its  $C^\alpha$  norm is estimated in terms of  $q_0$  and  $Q_0$  only.

(ii) Let  $\xi : \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}}$  be a Beltrami homeomorphism satisfying the equation

$$\xi_{\bar{z}} = q\xi_z, \quad \xi(1) = 1.$$

Since  $q \in C^{k,\beta}$ , then  $\xi \in C^{k+1,\beta}$  is a diffeomorphism (see [14], Theorem 2.10). By changing the independent variable to  $\xi$ , the equation (4) turns into

$$v_{\bar{\xi}} \bar{\xi}_{\bar{z}} (1 - |q|^2) = Q.$$

Since  $\xi \in C^{k+1,\beta}(\mathbb{D})$ , then the change of variable  $z \rightarrow \xi$  preserves the class  $C^{k,\beta}(\mathbb{D})$ . Hence

$$v = T_1 \frac{Q}{\bar{\xi}_{\bar{z}} (1 - |q|^2)},$$

here  $T_1$  is applied with respect to the variable  $\xi$ . It shows that  $v \in C^{k+1,\beta}(\mathbb{D})$ . It is clear from the construction that the norm of  $v$  is estimated in terms of  $\beta$ ,  $q_0$ , and  $Q_0$  only. The proof is complete. Q.E.D.

We consider some properties of almost complex structures with matrices of the form (3). We begin with coordinate changes that preserve this form.

**Proposition 2.2** *Let  $A$  be a complex matrix of a smooth almost complex structure  $J$  in a domain in  $\mathbb{C}^2$  with coordinates  $Z = (z, w)$ . Suppose  $A$  has the form (3) with  $|a| < 1$  and  $|c| < 1$ . Let  $Z' = (z', w')$  be a smooth orientation preserving coordinate change of the form  $z' = z$ ,  $w' = w'(z, w)$ . Then the matrix  $A'$  of  $J$  relative to the new coordinates also has the form (3). Furthermore, if  $w(z, 0) = 0$ ,  $a(z, 0) = 0$ , and  $b(z, 0) = 0$ , then for the corresponding entries of  $A'$  we have  $a'(z', 0) = 0$  and  $b'(z', 0) = 0$ . (Finally, if in addition  $w'_{\bar{w}}(z, 0) = 0$  and  $c(z, 0) = 0$ , then  $c'(z', 0) = 0$ .)*

**Proof :** We represent the coordinate change in an equivalent form  $z = z'$ ,  $w = w(z', w')$ . According to [13], the matrix  $A$  changes to

$$A' = (Z_{Z'} - A\bar{Z}_{Z'})^{-1} (A\bar{Z}_{\bar{Z}'} - Z_{\bar{Z}'}).$$

Then the matrix  $A'$  has the form (3) with entries

$$\begin{aligned} a' &= a, \\ b' &= (w_{w'} - c\bar{w}_{w'})^{-1} (b - a(w_{z'} - c\bar{w}_{z'}) + c\bar{w}_{\bar{z}'} - w_{\bar{z}'}), \\ c' &= (w_{w'} - c\bar{w}_{w'})^{-1} (c\bar{w}_{\bar{w}'} - w_{\bar{w}'}). \end{aligned}$$

Note that  $w_{w'} - c\overline{w}_{w'} \neq 0$  because the change of variables is orientation preserving and  $|c| < 1$ . The conclusions now follow from the above expressions of  $a'$ ,  $b'$ , and  $c'$ . Q.E.D.

For almost complex structures with matrix (3), one can reduce the Cauchy-Riemann system to a single equation.

**Proposition 2.3** *Let  $\zeta \rightarrow (z(\zeta), w(\zeta))$  be a  $J$ -complex curve for an almost complex structure with matrix (3), where  $|a| < 1$  and  $|c| < 1$ . Suppose  $z_\zeta \neq 0$ . Then the map locally can be represented by a graph of a function  $z \mapsto w(z)$  satisfying the equation*

$$w_{\overline{z}} = a_1 w_z + c_1 \overline{w}_{\overline{z}} + b_1, \quad (6)$$

whose coefficients are determined by  $A$ . The equation is elliptic, in particular  $|a_1| + |c_1| < 1$ . Moreover, if  $a(z, 0) = 0$  (resp.  $b(z, 0) = 0$ ,  $c(z, 0) = 0$ ), then  $a_1(z, 0) = 0$  (resp.  $b_1(z, 0) = 0$ ,  $c_1(z, 0) = 0$ ). Conversely, if  $z \mapsto w(z)$  satisfies (6), then its graph locally can be represented as a parametrized  $J$ -complex curve. Finally, the correspondence between the triples  $(a, b, c)$  with  $|a| < 1$  and  $|c| < 1$  and the triples  $(a_1, b_1, c_1)$  with  $|a_1| + |c_1| < 1$  is one-to-one.

**Proof :** By the Cauchy-Riemann equations (1),

$$z_{\overline{\zeta}} = a \overline{z}_{\overline{\zeta}}, \quad w_{\overline{\zeta}} = b \overline{z}_{\overline{\zeta}} + c \overline{w}_{\overline{\zeta}}. \quad (7)$$

By the Chain Rule,

$$w_\zeta = w_z z_\zeta + w_{\overline{z}} \overline{z}_\zeta, \quad w_{\overline{\zeta}} = w_z z_{\overline{\zeta}} + w_{\overline{z}} \overline{z}_{\overline{\zeta}}.$$

By eliminating  $z_{\overline{\zeta}}$ ,  $w_\zeta$ , and  $w_{\overline{\zeta}}$ , and canceling by  $\overline{z}_{\overline{\zeta}} \neq 0$  we obtain

$$w_{\overline{z}} - ac \overline{w}_z = -aw_z + c \overline{w}_{\overline{z}} + b. \quad (8)$$

Plugging (6) in (8) yields

$$a_1 - ac \overline{c}_1 = -a, \quad b_1 - ac \overline{b}_1 = b, \quad c_1 - ac \overline{a}_1 = c. \quad (9)$$

Solving (9) yields the coefficients of the desired equation (6):

$$a_1 = -a(1 - |c|^2)/\Delta, \quad b_1 = (b + ac \overline{b})/\Delta, \quad c_1 = c(1 - |a|^2)/\Delta, \quad \Delta = 1 - |ac|^2, \quad (10)$$

in which  $|a_1| + |c_1| < 1$ . Conversely, if  $w(z)$  satisfies (6), then we can find the parameter  $\zeta(z)$  by solving the linear Beltrami equation  $\zeta_{\overline{z}} + a(z, w(z))\zeta_z = 0$ , which is the first equation in (7) written for the inverse function  $z \mapsto \zeta(z)$ .

Finally, we observe that the equations (9) have a unique solution in  $(a, b, c)$  with  $|a| < 1$ ,  $|c| < 1$ . Indeed, put  $q = ac$ . Then (9) imply  $(a_1 - q \overline{c}_1)(c_1 - q \overline{a}_1) = -q$ . The latter has only one solution  $q$  with  $|q| < 1$  because  $|a_1| + |c_1| < 1$ . The equations (9) now give the triple  $(a, b, c)$  with  $|a| < 1$ ,  $|c| < 1$ . The other conclusions follow automatically. Q.E.D.

We need a maximum principle for solutions of (6). It is mentioned in [14] without proof. For completeness we include it here. Consider a linear equation

$$w_{\overline{z}} = q_1 w_z + q_2 \overline{w}_{\overline{z}} + Q_1 w + Q_2 \overline{w}, \quad (11)$$

**Proposition 2.4** *Let  $q_1, q_2, Q_1, Q_2$  be bounded functions in  $\overline{\mathbb{D}}$  and let  $|q_1| + |q_2| \leq q_0 < 1$ ,  $|Q_1| + |Q_2| \leq Q_0$ , here  $q_0$  and  $Q_0$  are constants.*

- (i) *There exists a constant  $C > 0$  depending only on  $q_0$  and  $Q_0$  so that for solutions of the equation (11) we have  $\max_{\mathbb{D}} |w| \leq C \max_{\partial\mathbb{D}} |w|$ .*
- (ii) *If a solution to the equation (11) satisfies  $\operatorname{Re} w|_{\partial\mathbb{D}} = 0$ , then either  $w$  does not vanish in  $\overline{\mathbb{D}}$  or  $w \equiv 0$ .*

We note that  $\|Q_1\|_{L^p} + \|Q_2\|_{L^p} \leq Q_0$  for some  $p > 2$  instead of  $p = \infty$  would suffice, but we do not need it here.

**Proof :** Let  $w$  be a solution of (11). Put  $q = q_1 + q_2 \frac{\overline{w_z}}{w_z}$  and  $Q = Q_1 + Q_2 \frac{\overline{w}}{w}$ . Then  $w$  also satisfies

$$w_{\overline{z}} = qw_z + Qw. \quad (12)$$

Let  $u$  and  $v$  be solutions of the Beltrami equations

$$u_{\overline{z}} = qu_z, \quad v_{\overline{z}} = qv_z + Q \quad (13)$$

so that  $u : \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}}$  is a homeomorphism. Such solutions exist because  $|q| \leq q_0 < 1$ . Then one can see ([14], Theorem 3.31) that every solution of (12), whence (11), has a representation

$$w(z) = \phi(u(z))e^{v(z)}, \quad (14)$$

where  $\phi$  is holomorphic in  $\mathbb{D}$ . Indeed, define  $\phi$  by (14) and plug in (12). Then using (13) we obtain  $(1 - |q|^2)\overline{u_z}e^v\phi_{\overline{u}} = 0$ . Hence  $\phi_{\overline{u}} = 0$ , and  $\phi$  is holomorphic.

Although the equations (13) depend on a solution  $w$ , by Proposition 2.1(i) there exists  $v$  whose sup-norm depends on  $q_0$  and  $Q_0$  only. Hence, the assertion (i) follows by the usual maximum principle applied to a holomorphic function  $\phi$ .

To prove (ii), we note that by Proposition 2.1(i), the solution  $v$  to the non-homogeneous equation in (13) can be chosen with the condition  $\operatorname{Im} v|_{\partial\mathbb{D}} = 0$ . Then  $\operatorname{Re} w|_{\partial\mathbb{D}} = 0$  implies  $\operatorname{Re} \phi|_{\partial\mathbb{D}} = 0$ . Since  $\phi$  is holomorphic, then  $\phi \equiv ic$ , where  $c \in \mathbb{R}$  is constant, and (ii) follows. The proof is complete. Q.E.D.

**Corollary 2.5** *Let the coefficients  $a_1$  and  $c_1$  of (6) satisfy  $|a_1| + |c_1| \leq a_0 < 1$  and let  $|(b_1)_w| + |(b_1)_{\overline{w}}| \leq b_0$ ; here  $a_0$  and  $b_0$  are constants. Suppose  $b_1(z, 0) = 0$ . Then there exists a constant  $C > 0$  depending only on  $a_0$  and  $b_0$  so that for every solution of (6) we have  $\max_{\mathbb{D}} |w| \leq C \max_{\partial\mathbb{D}} |w|$ .*

**Proof :** Our hypotheses on  $b_1$  imply that  $b_1(z, w) = Q_1(z, w)w + Q_2(z, w)\overline{w}$ , where  $|Q_1| \leq b_0$  and  $|Q_2| \leq b_0$ . Then the conclusion follows by Proposition 2.4(i). Q.E.D.

Finally we include a result on the regularity of the boundary value problem for (6).

**Proposition 2.6** *Let  $w \in W^{1,p}(\mathbb{D})$ ,  $p > 2$ , be a solution of (6) satisfying the boundary condition  $\operatorname{Re} w|_{\partial\mathbb{D}} = \phi$ . Suppose that  $|a_1| + |c_1| < 1$ ,  $a_1, b_1, c_1 \in C^{k,\alpha}(\mathbb{D} \times \mathbb{C})$ ,  $0 < \alpha < 1$ , and  $\phi \in C^{k+1,\alpha}(\mathbb{D})$ ,  $k \geq 0$ . Then  $w \in C^{k+1,\alpha}(\mathbb{D})$ .*

**Proof :** We interpret the graphs of solutions of (6) as  $J$ -complex discs and apply a reflection principle from [9]. According to it, if  $J$  is  $C^{k,\alpha}$  ( $k \geq 0$ ,  $0 < \alpha < 1$ ), then  $J$ -complex discs attached to a totally real submanifold are  $C^{k+1,\alpha}$ -smooth.

Let  $J$  be the almost complex structure corresponding to the equation (6) by Proposition 2.3. Then the solution of (6) with boundary condition  $\operatorname{Re} w|_{b\mathbb{D}} = \phi$  defines a  $J$ -complex disc attached to the totally real submanifold  $\{(z, w) : |z| = 1, \operatorname{Re} w = \phi(z)\}$ . The conclusion now follows by the reflection principle [9]. Q.E.D.

### 3 Proof of the main theorem

Without loss of generality we can assume  $\Lambda = b\mathbb{D} \times b\mathbb{D}$ . Indeed, otherwise we can transform the curves  $\{z\} \times \gamma_z$  into the circles  $\{z\} \times b\mathbb{D}$  by a smooth change of coordinates of the form  $z' = z$ ,  $w' = w'(z, w)$  so that  $w'(z, 0) = 0$  and  $w' = w$  for big  $w$ . By Proposition 2.2, the hypotheses of the theorem will be preserved by the change.

By Proposition 2.3, if a smooth  $J$ -complex disc in  $\overline{\mathbb{D}} \times \mathbb{C}$  can be represented as a graph of a smooth function  $w : \overline{\mathbb{D}} \rightarrow \mathbb{C}$ , then  $w$  satisfies the equation (6). The disc will be attached to  $\Lambda = b\mathbb{D} \times b\mathbb{D}$  if in addition  $|w(z)| = 1$  for  $z \in b\mathbb{D}$ .

Without loss of generality we can assume that the coefficients  $a$ ,  $b$ , and  $c$  have compact support in  $\overline{\mathbb{D}} \times \mathbb{C}$ . Indeed, by the maximum principle (Corollary 2.5), the boundary condition  $|w| = 1$  implies  $|w| < C$  for all  $z \in \mathbb{D}$ . Here  $C > 0$  depends only on the coefficients of (6). Then we can assume  $a = b = c = 0$ , hence  $a_1 = b_1 = c_1 = 0$  for  $|w| > C$ .

Since we are looking for a non-vanishing solution  $w$ , we now make a change of variable  $w = e^u$ . Then the equation (6) with boundary condition  $|w| = 1$  transforms into

$$u\bar{z} = a_2 u_z + c_2 \bar{u}_{\bar{z}} + b_2 \quad (15)$$

with boundary condition

$$\operatorname{Re} u(z) = 0, z \in b\mathbb{D}. \quad (16)$$

Here

$$a_2(z, u) = a_1(z, e^u), \quad b_2(z, u) = e^{-u} b_1(z, e^u), \quad c_2(z, u) = e^{\bar{u}-u} c_1(z, e^u).$$

The coefficients of (15) still have uniformly bounded derivatives in  $u$  and satisfy the ellipticity condition  $|a_2| + |c_2| < a_0 < 1$ . Under these hypotheses by [10] (pp. 335–351; see also [13], Proposition 4.2) for every  $z_0 \in b\mathbb{D}$  and  $\tau \in \mathbb{R}$ , the boundary value problem (15-16) has a unique solution with  $u(z_0) = i\tau$ .

Returning to the original variable  $w$ , we obtain a one-parameter family of discs attached to  $\Lambda$ . They are parametrized by  $\tau \in \mathbb{R}/2\pi\mathbb{Z}$  for fixed  $z_0$ , say  $z_0 = 1$ . Their boundaries are disjoint and cover all of  $\Lambda$ . Since  $w = e^u \neq 0$ , then they do not meet  $\overline{\mathbb{D}} \times \{0\}$ . By the continuous dependence statement of [10], the family is continuous. For the same reason, they depend continuously on  $A$ , which gives the continuous dependence conclusion in (ii).

The smoothness of the above one-parameter family follows by the implicit function theorem. Fix  $k \geq 1$ ,  $0 < \alpha < 1$ . By Proposition 2.6, every solution  $u \in C^{k+1,\alpha}(\mathbb{D})$ . Denote by

$L : C^{k,\alpha}(\mathbb{D}) \rightarrow C^{k-1,\alpha}(\mathbb{D})$ ,  $L : \dot{u} \mapsto L(\dot{u})$ , the linearization at  $u$  of the operator defined by (15). Then the operator  $L$  itself can be written in the same form with coefficients of class  $C^{k-1,\alpha}(\mathbb{D})$ . By Proposition 2.6 the linear map  $\mathcal{L} : C^{k,\alpha}(\mathbb{D}) \rightarrow C^{k-1,\alpha}(\mathbb{D}) \times C^{k,\alpha}(b\mathbb{D})$  defined by  $\mathcal{L}(\dot{u}) = (L(\dot{u}), \operatorname{Re} \dot{u}|_{b\mathbb{D}})$  is surjective. As it was shown above, the solution of the boundary value problem (15-16) is uniquely determined by the condition  $u(z_0) = i\tau$  for every  $z_0 \in b\mathbb{D}$  and  $\tau \in \mathbb{R}$ . Hence the kernel of  $\mathcal{L}$  is one-dimensional. By the implicit function theorem there exists a  $C^{k,\alpha}$ -smooth one-parameter family of solutions of (15-16). By uniqueness we obtain the smooth dependence conclusion in (ii).

The fact that the discs are disjoint follows by the positivity of intersections of  $J$ -complex curves. Indeed, we can include our boundary value problem  $|w| = 1$  for the equation (6) in a one-parameter family with boundary condition  $|w| = r$ . Let  $f$  be a  $J$ -complex disc constructed above, attached to  $\Lambda$ . Then we can construct a continuous family of  $J$ -complex discs  $f_r$  with boundary condition  $|w| = r$ , so that  $f_0$  is the disc  $\overline{\mathbb{D}} \times \{0\}$  and  $f_1 = f$ . Let  $g$  be another disc constructed above, attached to  $\Lambda$ . Since the boundaries of  $g$  and  $f_r$  do not intersect, then the intersection index of  $f_r$  and  $g$  is independent of  $r$ . Since  $g$  does not intersect  $f_0$ , then it does not intersect  $f_1 = f$  either.

We now prove that the surface  $\Gamma$  swept out by the discs is smooth. Let  $z \mapsto u(z, \tau)$  be the solution of (15) with conditions  $\operatorname{Re} u|_{b\mathbb{D}} = 0$  and  $u(1, \tau) = i\tau$ . It suffices to show that the map  $(z, \tau) \mapsto (z, u(z, \tau))$  is an immersion, which in turn reduces to  $\partial u / \partial \tau \neq 0$ . Put  $v = \partial u / \partial \tau$  and differentiate (15) with respect to  $\tau$ . Then we obtain a linear equation

$$v_{\bar{z}} = a_2 v_z + c_2 \bar{v}_{\bar{z}} + Q_1 v + Q_2 \bar{v} \quad (17)$$

with boundary condition  $\operatorname{Re} v|_{b\mathbb{D}} = 0$ . Here  $Q_1 = (a_2)_u u_z + (c_2)_u \bar{u}_{\bar{z}} + (b_2)_u$  and  $Q_2 = (a_2)_{\bar{u}} u_z + (c_2)_{\bar{u}} \bar{u}_{\bar{z}} + (b_2)_{\bar{u}}$ . Since  $v(1) = i \neq 0$ , then by Proposition 2.4(ii), the solution  $v$  does not vanish in  $\overline{\mathbb{D}}$  as desired.

Finally, the part (iii) follows by Corollary 2.5, the maximum principle for (6). The proof is complete. Q.E.D.

## References

- [1] B. Berndtsson and T. Ransford, *Analytic multifunctions, the  $\bar{\partial}$ -equation, and a proof of the corona theorem*, Pacific J. Math. **124** (1986), 57–72.
- [2] E. Bedford and W. Klingenberg, *Envelopes of holomorphy of certain 2-spheres in  $\mathbb{C}^2$* , J. Amer. Math. Soc. **4** (1991), 623–646.
- [3] M. Brunella, *Subharmonic variation of the leafwise Poincaré metric*, Invent. Math. **152** (2003), 119–148.
- [4] B. Coupet, A. Sukhov, and A. Tumanov, *Proper  $J$ -holomorphic discs in Stein domains of dimension 2*, Amer. J. Math. **131** (2009), 653–674.
- [5] J. Duval and D. Gayet, *Riemann surfaces and totally real tori*, arXiv 0910.2139.
- [6] F. Forstnerič, *Polynomial hulls of sets fibered over the unit circle*, Indiana Univ. Math. J. **37**(1988), 869–889.



- [7] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), 307–347.
- [8] Y. Eliashberg, *Filling by holomorphic discs and its applications*, Geometry of low-dimensional manifolds, **2** (Durham, 1989), 45–67, London Math. Soc. Lecture Series, 151, Cambridge Univ. Press, Cambridge, 1990.
- [9] S. Ivashkovich and A. Sukhov, *Schwarz reflection principle, boundary regularity and compactness for  $J$ -complex curves*, Ann. Inst. Fourier (Grenoble) **60** (2010), 1489–1513.
- [10] V. N. Monakhov, *Boundary problems with free boundary for elliptic systems of equations*, Translations of Mathematical Monographs, 57, Amer. Math. Soc., Providence, 1983, 522 pp.
- [11] A. Schnirelman, *Degree of a quasilinear map and the non-linear Riemann-Hilbert problem* (Russian), Matem. Sb. **89** (1972), 366–389.
- [12] Z. Slodkowski, *Holomorphic motions and polynomial hulls*, Proc. Amer. Math. Soc. **111** (1991), 347–355.
- [13] A. Sukhov and A. Tumanov, *Regularization of almost complex structures and constructions of pseudo-holomorphic discs*, arXiv 0809.4651, to appear in Annali Scuola Norm. Sup. Pisa.
- [14] I. N. Vekua, *Generalized analytic functions*, Pergamon, London-Paris-Frankfurt, 1962.